Math 564: Real analysis and measure theory
Lecture 4

Lebesgue premeasure on $\mathbb{R}^d$.

Analogously to Bernoulli measures on $A^{\mathbb{N}}$, we define a premeasure on the algebra $\mathcal{A}$ generated by boxes in $\mathbb{R}^d$. Note that the elements of $\mathcal{A}$ are finite disjoint unions of boxes, just like how the clopen sets in $A^{\mathbb{N}}$ for finite A are finite disjoint unions of cylinders.

We first define a premeasure on boxes:

$$\tilde{\lambda}(I_1 \times I_2 \times \dots \times I_d) := \mathrm{lh}(I_1) \cdot \mathrm{lh}(I_2) \cdot \dots \cdot \mathrm{lh}(I_d),$$

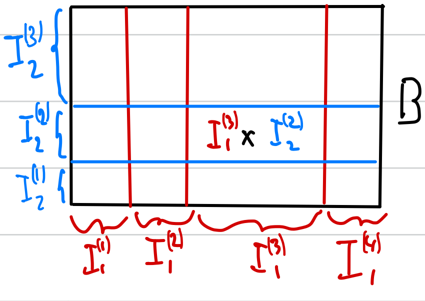
where $\mathrm{lh}(\text{interval}) = (\text{right endpoint}) - (\text{left endpoint})$ and we set $0 \cdot \infty = 0$.

We then "define" the potential premeasure on $\mathcal{A}$ by

$$\lambda(A) := \sum_{B \in \mathcal{P}} \tilde{\lambda}(B),$$

where $\mathcal{P}$ is any finite partition of A into boxes. As with Bernoulli, we need to show that this is well-defined, i.e. doesn't depend on the choice of the partition $\mathcal{P}$.

As with Bernoulli, we show that if a box is partitioned into a "homogeneous" collection of boxes, then $\tilde{\lambda}$ is finitely additive. The notion of "homogeneous" for boxes are grid-partitions, namely, a partition $\mathcal{P}$ of a box $B := I_1 \times I_2 \times \dots \times I_d$ into ones of the following form: each I_k is partitioned into finitely many intervals $I_k = \bigsqcup_{n < N_k} I_k^{(n)}$ and



$$\mathcal{P} = \{ I_1^{(n_1)} \times I_2^{(n_2)} \times \dots \times I_d^{(n_d)} : (n_1, n_2, \dots, n_d) \in N_1 \times N_2 \times \dots \times N_d \},$$

where we view $N := \{0, 1, \dots, N-1\}$.

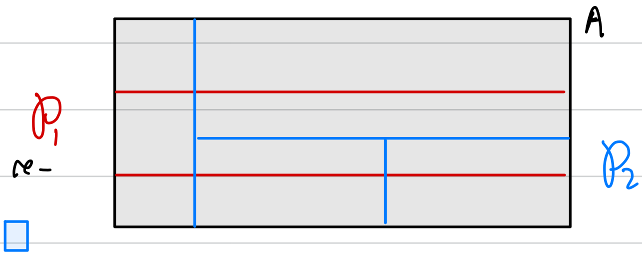
Claim (a). If $\mathcal{P}$ is a grid-partition of a box B, then $\tilde{\lambda}(B) = \sum_{P \in \mathcal{P}} \tilde{\lambda}(P)$.

Proof. This is trivial in $d=1$, and for $d>1$ apply induction using the distributivity law:

$$(a_1+a_2+\dots+a_k)\cdot(b_1+b_2+\dots+b_\ell)=\sum_{\substack{i\le k\\ j\le \ell}} a_i b_j. \qquad \square$$

Claim (b). If $\mathcal{P}_1, \mathcal{P}_2$ are two finite partitions of a set $A\in\mathcal{A}$ into boxes, then

$$\sum_{P_1\in\mathcal{P}_1}\tilde{\lambda}(P_1)=\sum_{P_2\in\mathcal{P}_2}\tilde{\lambda}(P_2)$$



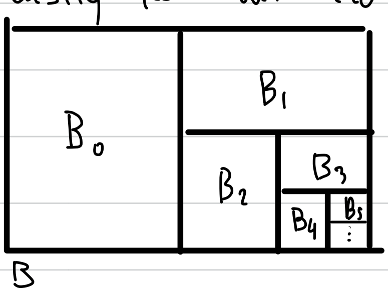
Proof. Take a grid-partition of A that is a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Details in HW. $\square$

This implies:

Cor. λ on $\mathcal{A}$ is well-defined and finitely additive.

Claim (c). λ is ctbly additive on $\mathcal{A}$.

Proof. Because a finitely additive measure is always ctbly superadditive, it's enough to prove ctbl subadditivity. We prove this in the case where a bounded box B is written as a ctbl disjoint union of boxes: $B=\bigsqcup_{n\in\mathbb{N}} B_n$. The general case follows easily from this and is left as a small HW exercise.



In case of cylinders of $2^{\mathbb{N}}$, we used that B is compact and the B_n are open, but for boxes neither is true in general. However we can approximate B by closed and B_n by open boxes. Fix an $\varepsilon>0$.

Notation. For $a,b\in\mathbb{R}$, we write $a\approx_\varepsilon b$ if $|a-b|\le\varepsilon$.

Let $B'\subseteq B$ be a closed box such that $\lambda(B')\approx_{\varepsilon/2}\lambda(B)$. Also for each $n\in\mathbb{N}$, let $\tilde{B}_n\supseteq B_n$ be an open box such that $\lambda(\tilde{B}_n)\approx_{\varepsilon\cdot 2^{-(n+1)}}\lambda(B_n)$. Then $\{\tilde{B}_n\}$ is an open cover of the compact set B', so there is a finite subcover $\{\tilde{B}_n\}_{n<N}$. Then:

$$\lambda(B) \approx_{\varepsilon/2} \lambda(B') \overset{\text{monot.}}{\leq} \lambda\Big(\bigcup_{n<N} \tilde{B}_n\Big) \overset{\text{finite subadd.}}{\leq} \sum_{n<N} \lambda(\tilde{B}_n) \leq \sum_{n\in\mathbb{N}} \lambda(\tilde{B}_n) \approx_{\varepsilon/2} \sum_{n\in\mathbb{N}} \lambda(B_n), \text{ so}$$

$\lambda(B) \leq \sum_{n\in\mathbb{N}} \lambda(B_n) + \varepsilon$, which implies $\lambda(B) \leq \sum_{n\in\mathbb{N}} \lambda(B_n)$ since ε is arbitrary. $\square$

We call this premeasure λ on $\mathcal{A}$ the lebesgue premeasure.

Carathéodory extension.

To define measures, we always define a premeasure on some algebra and apply the following theorem, where we call a premeasure μ on an algebra $\mathcal{A}$ on X σ-finite if $X = \bigsqcup_{n\in\mathbb{N}} A_n$ for some $A_n \in \mathcal{A}$ with $\mu(A_n) < \infty$.

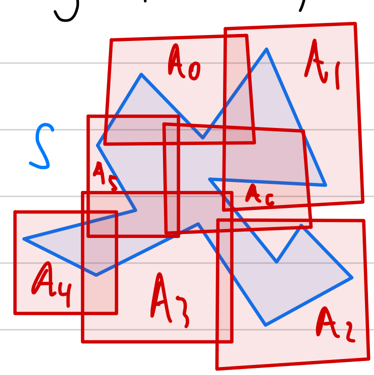
Theorem (Carathéodory). Every premeasure μ on an algebra $\mathcal{A}$ on a set X admits an extension to a measure on the sigma-algebra $\langle \mathcal{A} \rangle_\sigma$. Moreover:

- The outer measure μ^* is such an extension and any extension ν satisfies $\nu \leq \mu^*$.
- If μ is σ-finite, then the extension is unique and equal to μ^*.

To prove this, we need the following notion:

Def. Let $\mathcal{A}$ be an algebra on a set X and μ be a premeasure on $\mathcal{A}$. The outer measure of μ is the function $\mu^*: \mathcal{P}(X) \to [0,\infty]$ defined by: for $S \subseteq X$,

$$\mu^*(S) := \inf\Big\{\sum_{n\in\mathbb{N}} \mu(A_n) : \bigcup_{n\in\mathbb{N}} A_n \supseteq S \text{ and } \{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}\Big\}.$$



Properties of outer measure. (a) Monotone: $\mu^*(A) \leq \mu^*(B)$ for $A \subseteq B \subseteq X$.
(b) Ctbly subadditive: $\mu^*\Big(\bigcup_{n\in\mathbb{N}} S_n\Big) \leq \sum_{n\in\mathbb{N}} \mu^*(S_n)$ for all $S, S_n \subseteq X$.

Proof. (a) follows from the definition of μ^* because a cover of B is also a cover of A. Same for (b): the union of covers of the S_n is a cover of $\bigcup_{n\in\mathbb{N}} S_n$. $\square$

__Lemma.__ For any premeasure μ on an algebra $\mathcal{A}$, the outer measure μ^* on $\mathcal{A}$ is equal to μ:

$$\mu^*|_{\mathcal{A}} = \mu.$$

Proof. Let $A \in \mathcal{A}$ and let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a cover of A. By disjointification, we may assume that the A_n are disjoint; also by replacing A_n with $A_n \cap A$, we may assume $A = \bigsqcup_{n\in\mathbb{N}} A_n$. But then by ctbl additivity of μ, we have $\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n)$, so even with the original A_n, we had $\mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$ by monotonicity. $\square$

__Carathéodory's theorem: existence.__ Every premeasure μ on an algebra $\mathcal{A}$ on a set X admits an extension to a measure on the sigma-algebra $\langle \mathcal{A} \rangle_\sigma$. In fact, μ^* is such an extension.

Proof. To show that μ^* is ctbly additive on $\langle \mathcal{A} \rangle_\sigma$, it is enough to show that it is finitely additive because: outer measures are ctbly subadditive and finite additivity implies ctbl superadditivity.